A simple model of multiple equilibria and default

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Abstract

This paper presents a simple model that incorporates two types of sovereign default cost: first, a lump-sum cost due to the fact that the country does not service its debt fully and is recognised as being in default status, by ratings agencies, for example. Second, a cost that increases with the size of the losses (or haircut) imposed on creditors whose resistance to a haircut increases with the proportional loss inflicted upon them.

One immediate implication of the model is that under some circumstances the creditors have a (collective) interest to forgive some debt in order to induce the country not to default.

The model exhibits a potential for multiple equilibria, given that a higher interest rate charged by investors increases the debt service burden and thus the temptation to default. Under very high debt levels, credit rationing can set in as the feedback loop between higher interest rates and the higher incentive to default can become explosive.

The introduction of uncertainty makes multiple equilibria less likely and reduces their range.

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Introduction

It is now a commonplace to argue that a high level of public debt leads to indeterminacy for risk premia because even a rather high level of public debt could be sustainable if the government only had to pay the low interest rate corresponding to riskless investments. However, the same level of debt might become unsustainable, forcing a default, if the interest rate on public debt were much higher. Many authors have therefore argued that there might be multiple equilibria: if the market thinks the government can pay, it will be able to pay because the risk premium will be low. However, if the market thinks that the government cannot pay, it will not be able to pay, because the risk premium will be so high that debt service becomes too expensive for the government to finance. Doubts about the ability of a government to service its debt could thus become self-fulfilling. Kopf (2011) and de Grauwe (2011) have argued that within the euro area all national governments are in this situation and that a crisis could arise when the market just switches from the good (low interest rate) to the bad equilibrium (high interest rates), thus forcing a default.

The purpose of this paper is to present an analytical framework for this issue. The framework describes the problem facing a country that cannot print the currency in which its debt is denominated. This is the case for the euro area countries today and has been the case for the foreign debt of emerging market economies.

The model presented here is thus different in nature from the literature on speculative attacks on highly indebted countries, which looks at the trade-off between using the printing press or taxes to service high levels of public debt (see Obstfeld, 1995 and Adrian & Gros, 1999, for example). When the government cannot print the currency in which its debt is denominated, it has only the choice of increasing taxes or defaulting.

This choice between an increase in taxes or default is implicit in most of the literature on defaults and debt crisis that was motivated by the experience of the 1980s, when many emerging economies had problems servicing their (mostly foreign) debt. A common assumption in that literature, however, is that in some states there are simply too few resources for a country to service its debt and that in the case of default it just delivers to foreign creditors whatever foreign exchange earnings it has. A key feature of this paper is that the choice of how much to pay to creditors is a more gradual one, based on the cost of raising the necessary revenues and the cost these creditors can impose on the debtor country.

The key feature of the model presented is thus that the decision to default involves costs that have two quite different components:

i)   First, there is a threshold effect. The very fact that a government does not pay its debt fully and on time creates a 'credit event' which has serious costs for the government in terms of future market access, and usually also damages the market access for banks and enterprises of the country in question.

ii)   Second, once the threshold has been passed, there is still a trade-off between the amount the government is willing to pay and the taxes it has to raise to pay for this residual amount. Most defaults are followed by protracted negotiations between creditors and the government in which the creditors try to impose higher costs on the government if it pays less.

Section 1 outlines the basic set-up. Section 2 then analyses the feedback loop between interest setting and the incentive to default under certainty, i.e. when investors know exactly the circumstances under which a higher interest rate will increase the debt service burden to the point where the government will decide to default, thus validating the high interest rate set by the market. Section 3 introduces uncertainty and shows that the parameter range for which multiple equilibria are possible is reduced if the decision rule concerning default is not fully deterministic. Under uncertainty, the positive feedback loop between debt service burden and default is not certain and hence not so strong. Section 4 illustrates one special case for which uncertainty can eliminate the potential for multiple equilibria. Section 5 concludes.

1.   The model

This model describes the problem facing a government that has to decide during the present, final period whether or not to service its debt in full. At this point the debt level is simply inherited from the past.

The starting point is a conventional loss function, which expresses the idea that increasing tax revenues leads to increasing distortions and that these distortions become more costly as the tax rate increases. The social loss arising from obtaining tax revenues measured as a percentage of GDP is thus assumed to be given by:

$$(1) \quad L = \alpha q^2 \quad \alpha > 0$$

where q indicates the ratio of tax revenues to GDP, or the overall effective tax rate, and the parameter α measures the inefficiency of the tax system, a higher value implies a lower efficiency.

If the government chooses to default it also has to deal with the opposition of its creditors, which will be stronger the higher the haircut imposed on them and the larger the debt on which this haircut is imposed. The cost of default thus has two components: a lump sum once this option has been chosen and then a variable cost which increases with the size of the total losses imposed on creditors.

This is captured by extending the social loss function in the following way:

$$(2) \quad L_d = \propto q^2 + dh^2 + LL_d$$

Where L_d now represents the total social loss if the government has chosen to default.

This social cost has three components:

i)   There is the lump-sum cost of entering into the default mode (credit rating agencies putting the rating of the country on 'default'), which in general occurs immediately if the country does service all of its public debt (denoted by d) in full and on time. This lump-sum cost is denoted by LL_d (>0).

ii)   Choosing the default option does not mean that nothing is paid on debt. On the contrary, the government still has to decide what fraction to pay, or equivalently, what 'haircut', denoted by h, to impose on creditors. The resistance from creditors will be

stronger the greater their total losses. This resistance should be proportional to the amount of debt, denoted by d, on which the haircut applies; a higher haircut will encounter increasing resistance.

iii) Finally there is still the cost of raising revenues from taxes to pay for the residual debt service, which create distortions and thus social costs, αq^2.

Lower case d thus represents the debt-to-GDP ratio and h is the 'haircut' imposed on creditors. This notation implies that the proportion of the total debt the government is willing to pay (if it has chosen the default option) would be (1-h). Creditors will of course resist any haircut and try to inflict costs on the government (e.g. law suits, attachment of assets, etc.) and their resistance should be stronger the larger the cut they have to take. Moreover, the political cost in terms of pressure from the governments of creditors' home countries is likely to increase the larger the haircut.

The parameter α represents the cost arising from higher taxes relative to the intensity of the costs the creditors can impose on a government that wants to impose a haircut on them.

If government debt is owed to foreign residents the parameter α should be higher because the political cost of taxing voters to transfer resources to foreigners will certainly be perceived to be higher than if the government has to levy taxes whose proceeds are paid out to other residents, and thus voters.

The loss function specified in equation (2) implies that there is a trade-off between increasing the 'haircut' imposed on creditors and increasing taxes to service the debt. As this paper concentrates on the choice between taxation and debt service all other expenditure is neglected and it is assumed that taxes need only to be raised to pay off the existing debt, d. This implies immediately that the economy-wide tax rate (or equivalently the primary surplus) required to pay off its debt in full is equal to the debt (or rather the debt-to-GDP ratio): $q = d$. It follows that if debt is serviced in full the social loss is given simply by:

$$(3) \quad L_f = \alpha d^2$$

If the government chooses the default option it still has to make a decision as it has to decide on the size of the haircut. This is in reality an important decision since no country (except in extreme circumstances, like war) has just declared a default and then paid nothing. In reality the declaration (or rather admission) of a government that it is not able to pay its creditors in full is usually followed by a period of negotiation over the proportion of its debt the government feels able to pay.

A key consideration in this context is that the tax rate necessary after a default is determined by the size of the haircut since the remaining debt obligation will amount to (1-h)d. This implies that the social loss in case of default is determined by the solution to the problem:

$$(4) \quad \min_h L_d = \alpha((1-h)d)^2 + dh^2 + LL_d$$

This implies that the size of the haircut, h, which minimises the social loss, will be given by:

$$(5) \quad -\alpha 2(1-h)d^2 + 2dh = 0 \implies h = \frac{\alpha d}{(1+\alpha d)} < 1$$

The haircut as a fraction of the value of debt thus depends on the level of debt to be serviced and the cost of raising tax revenue (relative to the costs the creditors can inflict on the government). The higher this cost, i.e. the higher α, the more attractive it becomes for the government to reduce the need to have high taxes to service the debt, which implies that a

higher haircut becomes more attractive. A higher level of debt also makes a higher haircut desirable, as one would expect.

The solution for the haircut can be substituted back into the loss function to calculate the loss under default:

$$(6) \quad L_d = \alpha d^2 \left[1 - \frac{\alpha d}{(1+\alpha d)}\right]^2 + d \left[\frac{\alpha d}{(1+\alpha d)}\right]^2 + LL_d$$

which can be simplified to:

$$(7) \quad L_d = \alpha d^2 (1 + d\alpha)^{-1} + LL_d$$

The difference between the loss under full debt service and the loss under default is thus given by (see equations (3) and (7)):

$$(8) \quad L_f - L_d = \alpha d^2 - \alpha d^2 (1 + d\alpha)^{-1} - LL_d = \alpha^2 d^3 (1 + d\alpha)^{-1} - LL_d$$

The first part of this difference ($\alpha^2 d^3/(1+\alpha d)$) is always positive because imposing a haircut on creditors allows the government to reduce costly taxes by operating on another margin (defaulting on its creditors). This part might be called the 'temptation' because it represents the gain in terms of lower welfare costs the government could accrue through a default.

The temptation has to be set against the lump-sum cost of defaulting. The entire expression on the right-hand side of equation (8) could of course be negative if the lump-sum cost of defaulting is high enough. In this case default would not be a rational option. But the opposite is also true: if the loss from servicing debt in full is larger than under default it would be rational for the government to default. Condition (8) thus defines implicitly the threshold for the debt level (given the lump-sum cost of defaulting and the relative cost of raising tax revenues) at which the government will just be induced to default. A country with a debt level above the threshold defined by condition (8) could be said to have a debt overhang.

Inspection of condition (8) shows that the maximum debt the country is willing to service fully is negatively related to the parameter α. This is what intuition would suggest: the higher the cost of raising tax revenues, the lower the debt carrying capacity of the country.

A first result of this simple setup concerns the relationship between the level of debt and what the government will actually pay, which should determine the market value of its debt.

In models without specific costs of default the market value of the debt of the country increases one-to-one with the nominal value of the debt. It is then simply capped at the ceiling determined by the exogenously given debt service capacity of the country. In this set-up, however, the value of debt will increase with the level of debt up to the limit given by equation (8) which describes the default threshold.

Figure 1. How much will be paid?
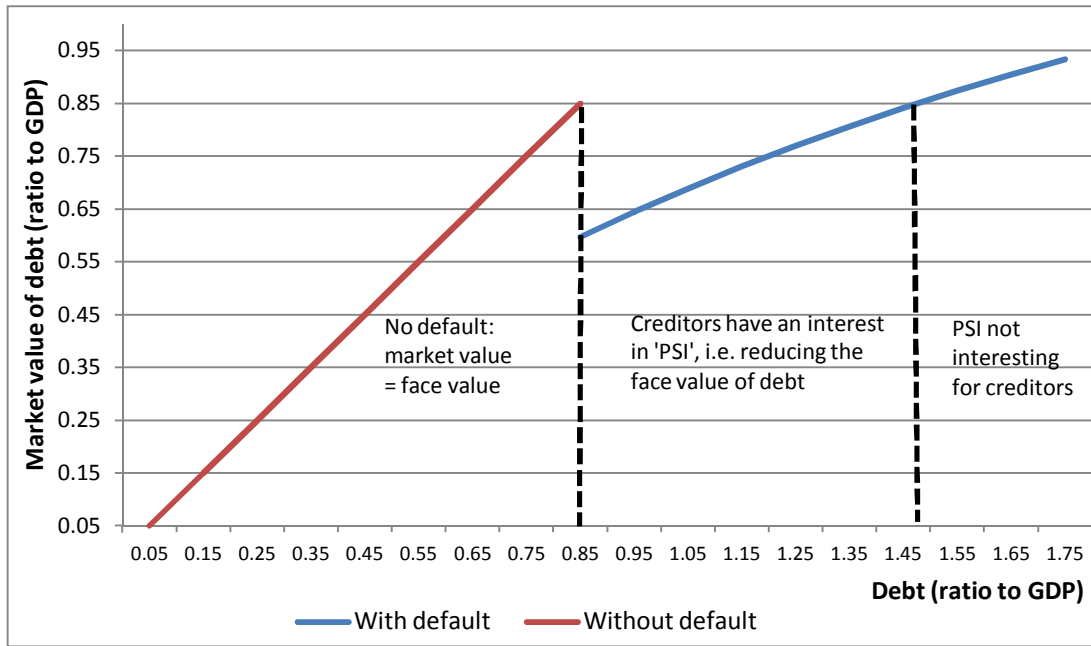
Note: The two lines are based on α=0.5 and LL$_d$ (=0.1) such that default occurs when d=0.9.

Figure 1 shows that to the right of the point determined by the condition that the social loss under default is equal to the loss under no default, the value of the debt jumps downwards, because with a slightly higher debt the government will prefer to default and impose the haircut determined by equation (5) and the government will pay out only

$$(9) \quad (1-h)d = \frac{1}{(1+\alpha d)}d < d$$

The relationship between the debt level and what the government would be willing to pay (under default) is not linear because a higher debt level leads to a higher haircut. Equation (9) implies that the maximum the government would be willing to pay (as a proportion of GDP) as the debt level goes to infinity is equal to 1/α.

Figure 1 shows the relationship between the level of debt (on the horizontal axis) and what the government will actually pay (vertical axis) for a parameter constellation (α=0.5), and a level of the lump-sum cost of default, which implies that the threshold at which the government will default is equal to 85% of GDP.

One immediate implication is that the creditors have an interest in unilaterally forgiving some debt if the debt is just above the threshold that will trigger a default. Krugman (1988) also arrives at the conclusion that above certain levels of debt it might make sense for creditors (if they can act collectively) to unilaterally forgive some debt. In Krugman (1988) his result requires a 'Laffer curve' for tax revenues, whereas in the model presented here the result comes about because of the lump-sum cost of default.

The existence of default costs thus generates a collective action problem: if the debt level is somewhat above the level at which it would be rational for the country to default, the collective interests of all creditors would be served best by forgiving just enough debt to

bring the country back to a situation where 'default does not pay'. However, each individual creditor would of course like to have his individual claim served in full.[1]

The case of Greece, which agreed on a 'voluntary' reduction of its debt with its private creditors under a complex operation called officially 'Private Sector Involvement' or PSI, could be understood in this way. Through the PSI operation, which was implemented in early 2012, Greece avoided a full-scale default; and maybe its private creditors did receive more than they would have received if Greece had gone into a formal default.

However, debt forgiveness does not make sense above a certain threshold level of debt. As one can see also from Figure 1, at very high levels of debt, creditors would be better off if the country goes into default: even with a 'haircut', they would receive more than the maximum amount the country would be willing to pay in order to avoid default. This will be the case for debt levels higher than the upper limit of the middle panel of Figure 1. The upper limit of this region can be determined from the condition that the payout to creditors is equal to the maximum debt level at which default is still not chosen. Designing the latter as d_d and the former as $d_{no\ PSI}$, condition (9) yields:

$$(10)\quad \frac{d_{no\ PSI}}{(1+\alpha d_{no\ PSI})} \geq d_d$$

The maximum proportional haircut that would be in the (collective) interests of creditors is thus given by:

$$(11)\quad \frac{d_{no\ PSI}-d_d}{d_{no\ PSI}} = \alpha d_d$$

The haircut that makes creditors indifferent between agreeing to a collective debt reduction or seeing the country go into default is thus given by the product of the cost of raising tax revenues (α) and the debt level that is just sustainable without default (which is a function of the same parameter and the lump-sum cost of default).

This completes the description of the final period during which what is left to decide is whether to service its debt in full and what the size of the haircut will be if default is chosen. The key question is of course how a situation with a debt overhang could arise. Why would investors lend the government more money than it would choose to repay? One answer could be that the total amount of debt was not publicly known, as in the case of Greece. Another possibility is that new information emerges about the capacity of the country to collect revenues (again, one could interpret the Greek experience in this way). But another possibility is that the final-period debt level is too high because interest rates were fixed at a high level during the previous period. This mechanism is explored in the next section.

2. Interest rate setting and multiple equilibria

The decisive action takes place during the previous period, during which the debt level to be serviced in the final period is determined by the interaction between the debt level inherited from an earlier period and the interest rate set in financial markets. Abstracting again from primary expenditure, the debt level to be serviced in the final period is determined by the debt level at the beginning of the previous period and the interest rate on public debt determined during this period:

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[1] An extension that is not pursued here is to consider the interests of the country, which would of course benefit from a larger debt forgiveness and could thus threaten to default strategically. Analyzing such a set-up would require the application of the tools of game theory (see also Tirole, 2012).

$$(12) \quad d = (1 + i)d_{-1}$$

Where d_{-1} denotes the debt level of the previous, starting, period and i denotes the interest rate set in the market.

The setting of this interest rate is a key aspect of the model because this is what gives rise to the potential for multiple equilibria. Investors are aware that the government could default in the final period and that they might be subject to a haircut. In this section it is assumed that the decision rule of the government is simply to calculate the difference between the social loss under default and full payment and then to implement whatever gives rise to a lower social loss.

The market for government debt is determined by a simple arbitrage condition. Investors have an alternative investment in the form of a riskless asset, yielding the rate ρ.

This implies that investors would be willing to buy the debt of the government in question at this riskless rate if, and only if, they assume that there will be no default. If a default is expected, the interest rate must of course be higher to compensate for the haircut investors can expect (with certainty).

The model can best be solved backwards by first analysing the case in which no default is expected.

Conditions for no default equilibrium

In this case the (expected) haircut is zero and the interest rate is given by:

$$(13) \quad 1 + \rho = 1 + i$$

This can be an equilibrium if under this interest rate default 'does not pay', i.e. if the social loss from default is indeed larger than under full payment. Using the values for the final period debt level given by (10) and (11) in expression (8) for the difference between the loss between default and no default yields:

$$(14) \quad L_f - L_d = \alpha^2 \big(d_{-1}(1 + \rho)\big)^3 (1 + d_{-1}(1 + \rho)\alpha)^{-1} - LL_d < 0$$

To simplify the notation the variable $d_{-1}(1+\rho)$ will henceforth denoted be by D:

$$(15) \quad D \equiv d_{-1}(1 + \rho)$$

The variable D could be called the 'risk-free debt' since this is the debt level the country would have in the final period if it could finance its debt at the risk-free rate. Since this parameter is the product of the inherited debt level and the risk-free interest rate (or the alternative rate of return accepted by investors), it follows that a shock to global interest rates could have the same effect within this model as a higher debt level.

The condition for full debt service to be the rational option can thus be written as:

$$(16) \quad L_f - L_d = \alpha^2 D^3 (1 + D\alpha)^{-1} - LL_d < 0$$

or

$$(17) \quad [1 + D\alpha]^{-1} \alpha^2 D^3 < LL_d$$

The left-hand side of this equation shows the gain from defaulting, the right-hand side the lump-sum cost of doing so. Full debt service becomes a self-fulfilling prophecy if the

combination of the initial debt level (increased for the riskless interest rate) and the cost of extracting taxes from the economy (α), is low enough to be smaller than the threshold determined by the lump-sum cost of default. Full debt service is never rational if this lump-sum cost is zero. However, the converse also holds: if this lump-sum cost is positive, full debt service is rational if the debt and/or the cost of taxation is low enough.

Conditions for default equilibrium

If investors expect a default, the domestic interest rate must be given by the riskless rate plus a term that reflects the fact that under default creditors get paid only a fraction (1-h) on every unit they are due:

$$(18) \quad 1 + \rho = (1 + i)(1 - h)$$

This can be rewritten more conveniently in terms of the 'risk premium', i.e. the differential between the riskless rate and the rate actually paid by the government. Strictly speaking there is no risk when there is no uncertainty. Under certainty one should thus call it a 'default' premium. This difference between the default and no-default interest rate can be defined as:

$$(19) \quad r \equiv \frac{1+i}{1+\rho}$$

The market equilibrium interest rate under default can then be written as:

$$(20) \quad r = (1-h)^{-1}$$

Given that the optimal choice of h, given default, is equal to αd/(1+αd) (see equation (5)) this implies that in this case the risk premium would be given by:

$$(21) \quad r = \left(1 + d_{defexp}\alpha\right)$$

where the final period debt level, if default is expected, is denoted by d_{defexp}. This final period debt level is of course influenced by the interest rate set in the market and thus the risk premium. Given the definition of D this can be written as:

$$(22) \quad r = (1 + Dr\alpha)$$

which implies that the risk premium becomes a function of the cost of obtaining tax revenues and the 'risk-free' debt level:

$$(23) \quad r = (1 - \alpha D)^{-1}$$

Equation (18) can be used to show that in this case (default expected) the haircut would be equal to:

$$(24) \quad h = \alpha D$$

It is apparent that the last two equations are defined only over the range 0≤αD<1. A combination of too high a debt level with high costs of levying taxes would lead to interest rates going to infinity (the haircut going to unity) as the product αD approaches 1. When this borderline case is reached, the country would experience credit rationing in the sense that it would not be able to obtain financing under any conditions (always under the condition that a default is expected ex ante).

This implies that a country might not be able to obtain financing at any rate if some news about either α or D emerges. For example, a sudden increase in global interest rates might increase D or news about disappointing tax revenues might increase the perceived value of α.

However, provided the condition (0≤αD<1) is met, the country might still obtain financing even if default is expected. The expectation of default can constitute an equilibrium if, with the interest rate fixed in the expectation of default, the government would indeed choose to default, i.e. if the social loss from paying in full is larger than the loss from default:

$$(25) \quad \alpha Dr(Dr)^2[1 + \alpha Dr]^{-1} > LL_d$$

This can be simplified to:

$$(26) \quad D\left[\frac{\alpha D}{(1-\alpha D)}\right]^2 > LL_d$$

Inspection of inequality (26) also shows immediately that the condition for a default equilibrium to be ex-post validated cannot be satisfied if there is no debt (i.e. if d_{-1} or D= 0). In this case there can be no feedback between higher interest rates and a higher final period debt level. The same also holds if the cost of raising tax revenues goes to zero. Economically this means that the final period debt might be high, but if it can be financed without a large social loss the temptation to default will remain low. This is what one observes in reality. Countries with low debt or a high ratio of tax revenues to GDP, which generally indicates low collection costs, enjoy lower interest rates.

In a model without uncertainty, it makes no sense to speak of a risk premium, but the difference between the riskless interest rate and the higher rate often paid by countries with high debt levels is called a risk premium. The next section will show how in a model with uncertainty the risk premium could be a smooth function of debt levels and other parameters of the model.

Multiple equilibria

Comparing conditions (17) and (26) shows immediately that they overlap given that the left-hand side of condition (17) is clearly smaller than the left-hand side of condition (26). There is a region for the lump-sum cost of default to be high enough for full debt service equilibrium to be chosen if the interest rate is low, but at the same time not high enough to make a default unattractive if the interest rate set in the market has been set in anticipation of default.

$$(27) \quad D\left[\frac{\alpha D}{(1-\alpha D)}\right]^2 > LL_d > D\frac{\alpha^2 D^2}{(1+D\alpha)}$$

It follows that within the region defined by the two inequalities in (25) both equilibria are possible: default and no default. Which equilibrium materialises cannot be determined in this model. Outside this region only one equilibrium is possible: if the lump-sum cost is higher than the left-hand expression of the condition (25) (or alternatively with very low debt and collection costs), the default risk is zero because in this case full repayment would always be chosen even if the market had expected default and the debt of the country is really 'risk-free' in the sense that the only possible equilibrium is that the rate of interest is low and the country services its debt in full.

Conversely, if the cost of default is lower than the right-hand side of expression in condition (27) (alternatively if the debt level is very high), the only consistent equilibrium is the one with default (and the high risk premium is justified).

Figure 2. Multiple equilibria under certainty
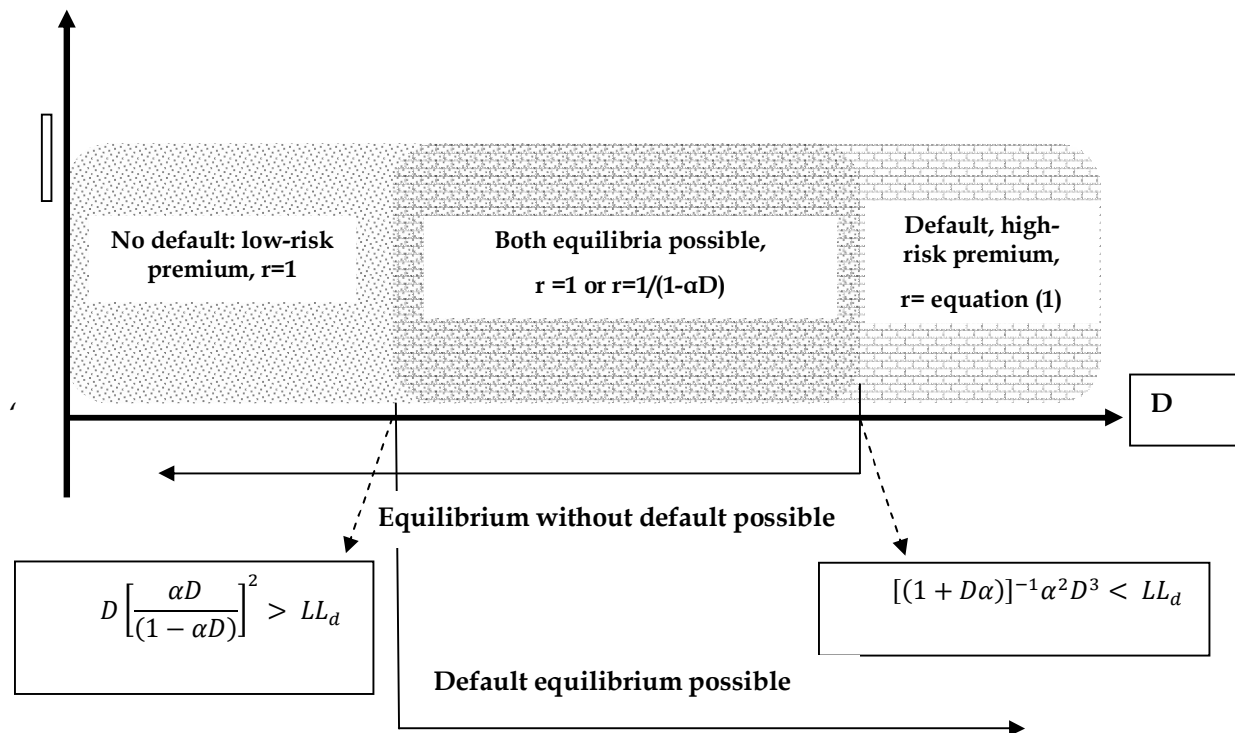
The region where default could occur, but where no default is also an equilibrium is illustrated in Figure 2 for increasing values of the debt-to-GDP ratio evaluated at the risk-free rate, D.

It is apparent from (27) that a higher debt level increases the lump-sum cost of default required to make default unattractive. For example, one could argue that for member countries of the euro area the lump-sum cost of defaulting is higher in political terms and possibly also in economic terms because of the higher degree of economic and political integration, thus making the no-default equilibrium more likely. This argument was used during the 1990s during the drive towards EMU, as it appeared at the time that the much higher degree of commitment implicit in EMU membership would lower risk premia.

The equations also show that a higher cost of raising taxes, i.e. a higher value of α, will have the same effect as a higher debt level, making, *ceteris paribus*, a default more likely.

It is also apparent that if the lump-sum cost of defaulting were zero, the country would always default (provided D is positive) since it would always gain at least a small advantage from hair-cutting creditors. This might explain why the debt of countries that just went through a default trades at high-risk premia: the lump-sum cost of a second default in a short time is presumably much lower than that of the first one. A second default is difficult to exclude unless debt has been cut to a very low level.

The width of the region inside which both equilibria are possible is determined by the lump-sum loss of defaulting (LLd) and the cost of raising tax revenues. Formally it is given by the difference between the upper and lower limit in the inequalities (27):

(28) *Size of Region of* LL_d *with multiple equilibria* $= D\left[\dfrac{\alpha D}{(1-\alpha D)}\right]^2 - D\dfrac{\alpha^2 D^2}{(1+D\alpha)} = D\alpha D\left[\dfrac{\alpha D}{(1-\alpha D)}\right]^2$

The region for the debt level in which both equilibria are possible becomes wider the higher the debt level (in a highly non-linear way) and the higher the cost of collecting taxes. As mentioned above, credit-rationing sets in when the product αD exceeds unity.

From a policy perspective the key question is whether there is a case for avoiding the higher risk premia which lead to the 'default equilibrium'. Investors expect the same return under both equilibria; otherwise interest rates would move. This implies that ex ante they should be indifferent on this account.

However, for the country concerned welfare is clearly lower under the default equilibrium because of the social cost of default. Payments to creditors are the same in both cases (again because this is necessary to make investors buy the bonds of the country). The cost of default is thus the net cost for the country of a higher risk premium.

Applied to the euro crisis these results imply that the debtor countries would benefit from any mechanism that limits risk premia on their public debt. The simple set-up of this model does not permit one to analyse the interests of other actors (see Tirole, 2012) in the interplay between interest of creditor and debtor countries). However, it is clear that a country that perceives the potential for multiple equilibria should be willing to make some side payment to induce others to create some mechanism to keep the risk premium to the 'no-default' equilibrium.

In a set-up without uncertainty the 'no default' equilibrium actually implies that the country has to pay only the risk-free interest rate. This is of course an extreme case, since in reality it is never certain that a country will be able to fully service its debt (and, conversely it is also rarely certain that a country will default). The next section therefore enriches the model in this direction.

3.   Uncertainty

With a binary decision rule for the government there is no uncertainty once the interest rate has been set. However, in reality this is not the case. Even once the interest rate has been set, much uncertainty persists. Even if the final period debt to be serviced can then be calculated rather precisely, nobody can be certain about whether there will be a default or not.

One key reason for this is that the time period considered here is one of several years, with the final period encompassing the entire future. Many things could happen between the time the interest rate is set and the time the decision on default is taken. One particularly important aspect is the political process. In reality at least one, but possibly several elections are likely to be held between the current period (in which interest rates are set in the market) and the final period (when the government decides whether to default and what haircut to impose on creditors). Different parties will compete and might have different views on default. Some of them might oppose default on non-economic grounds, or because they have a different view of the lump-sum cost of defaulting.

It is thus not enough for the loss under full debt service to exceed the loss under default to always trigger a decision to default.

When investors consider the price at which to buy the bonds of the government in question they all know that under certain conditions default is a possible outcome. Denoting the probability that the government might default with π, the setting of the interest rate will be determined by the expected return, which in turn is given by two elements:

- With probability (1-π) the government does not default. In this case the investor gets the full return (1+i).

- If the government does opt for default (with associated probability π) the investor gets only (1-h)(1+i).

The interest rate accepted by risk-neutral investors who have an alternative rate of return equal to the riskless rate ρ will thus be given by:

$$(29) \quad 1 + \rho = (1 - \pi)(1 + i) + \pi(1 + i)(1 - h)$$

or

$$(30) \quad 1 + i = (1 + \rho)/(1 - \pi h)$$

A consistent solution to the model requires of course that the probability of default assumed by investors is equal to the one that will result given the level of debt in the final period, which in turn is determined by the interest rate set in the previous period.

The equilibrium condition (29) can be rewritten in terms of the risk premium, r. Keeping in mind that the haircut is also a function of r – equations (5) and (14) – this yields:

$$(31) \quad r = \frac{1}{\left(1 - \pi \frac{\alpha Dr}{1 + \alpha Dr}\right)}$$

As the probability can only vary between 0 and 1, this implies that the risk premium can vary only between 1 (equal to the no 'default with certainty' equilibrium discussed above) and (1+αDr), which is equivalent to the 'default is certain' equilibrium found above.

The key issue is what factors influence the probability as perceived by investors that the default option will be chosen.

In reality the decision to default will be determined in a complex political process in which many factors interact, including external pressure and perceptions of how a default will impact the future capital market access of the country. The interplay of all these factors is difficult to disentangle ex ante; however, it is likely that the party(ies) that favour(s) default will have a greater chance of winning out in the political process the larger the difference between the welfare loss under default and the welfare loss under full debt service. A bigger difference in the loss should thus make default more likely.

It was established above that a higher risk premium increases the debt service burden, thus increasing the attractiveness of default. In general one could thus write:

$$(32) \quad \pi = \pi(\Delta L(r)) \text{ for } \Delta L \equiv \left(L_f - L_d\right) > 0 \text{ and with } 0 < \pi(.) < 1, \pi' > 0,$$

If this relationship is recognised in the determination of the risk premium, it implies that there is a feedback loop between a higher risk premium and a higher probability of default, which can be made explicit by substituting the determinants of the probability of default (32) into the market equilibrium condition (31).

$$(33) \quad r = \frac{1}{\left(1 - \pi(\Delta L(r)) \frac{\alpha Dr}{(1 + \alpha Dr)}\right)} = \frac{1 + \alpha Dr}{1 + \alpha Dr[1 - \pi(\Delta L(r))]}$$

One can now envisage that a consistent equilibrium is established in a *tâtonnement* process: The right hand side of this equation can be viewed as the risk premium investors would require if the initial attempt to find equilibrium were equal to the left-hand side. Equilibrium

can be obtained only when the two are equal. As long as the risk premium required by investors is higher than the left-hand side the auctioneer would have to increase the rate.

Given that the range for the risk premium is given by [1, αD] (with the restriction αD<1), it is sufficient to start by comparing the two extreme values of the risk premium to establish that an interior equilibrium exists at which default is neither ruled out nor certain.
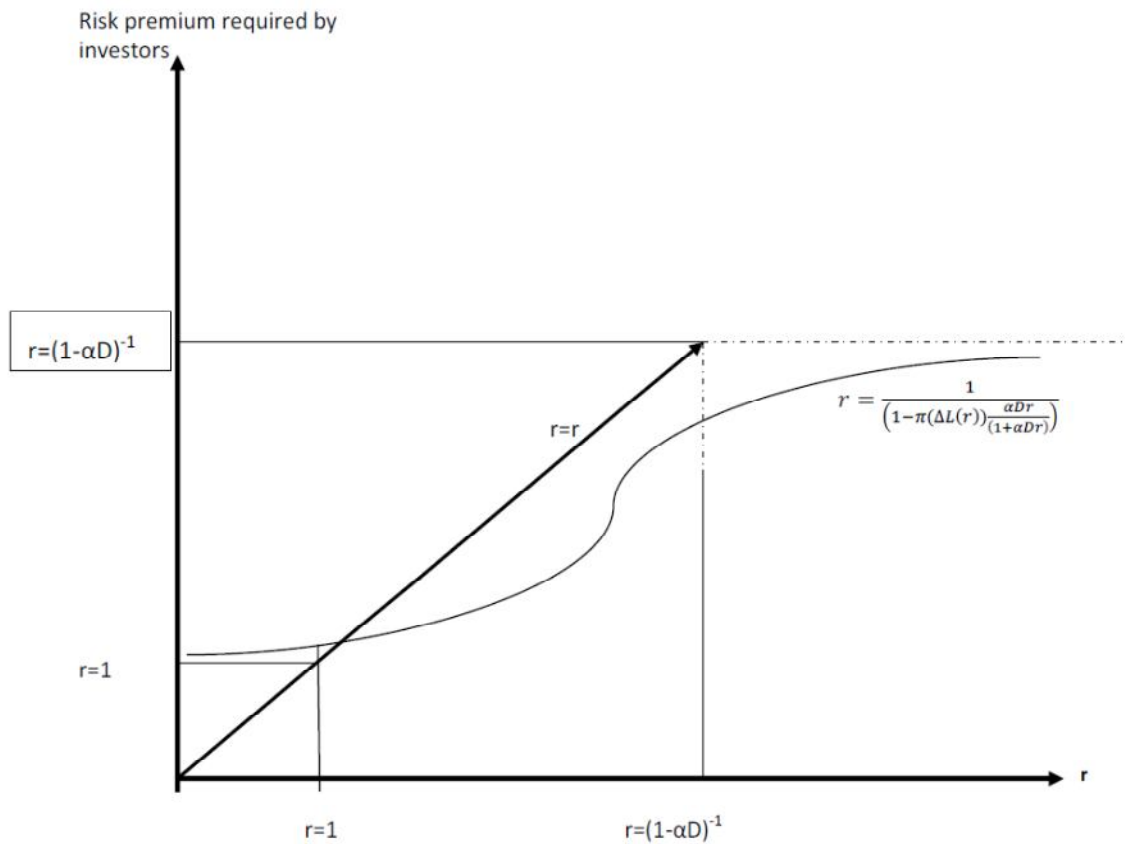
The lowest possible value for r is 1. At this value the right-hand side of equation (33) will clearly be greater than 1 as long as the probability of default is not strictly zero, even if the country does not have to pay a risk premium.

$$(34) \quad \frac{1}{\left(1-\pi(\Delta L(r=1))\frac{\alpha D}{(1+\alpha D)}\right)} < 1$$

This implies immediately that r=1 does not constitute a consistent equilibrium as the investors will require some risk premium, however small, to be compensated for the probability, even if remote, that a default might occur. A model-consistent equilibrium will then require r>1.

Figure 3 below illustrates this case. In this figure the right-hand side of equation (33) has been drawn in such a way that the probability of default goes to 1 only as r goes towards infinity and the probability of default goes to zero only as r also goes to zero (r= zero would mean zero debt service).

Figure 3. Single equilibrium under uncertainty

The largest possible value for r is equal to r=(1-αD)$^{-1}$ (if default is certain). However, in this case the right-hand side of equation (31) is smaller than this value as long as the probability of a default is strictly less than 1 even if r=(1-αD)$^{-1}$. This can be seen by substituting r=(1-αD)$^{-1}$ into equation (33) and simplifying:

$$(35) \quad r = \frac{1}{\left(1-\pi(\Delta L(r=1/(1-\alpha D)))\frac{\frac{\alpha D}{(1-\alpha D)}}{(1+\alpha D/(1-\alpha D))}\right)} = \frac{1}{[1-\pi(\Delta L(r=1/(1-\alpha D)))\alpha D]} < \frac{1}{1-\alpha D}$$

The risk premium associated with certain default (r=(1-αD)$^{-1}$) cannot thus constitute an equilibrium. Again, see Figure 3 for an illustration.

Taken together these two conditions imply that the equilibrium risk premium would have to lie between the two extremes r=1and *r=(1-aD)$^{-1}$*.

These considerations imply that some general properties of the equilibrium can be established without determining the details of the link between the probability of default and the social loss.

A first result is that uncertainty reduces the range of the equilibria possible if default becomes certain or impossible only outside the limiting values of the risk premium under certainty. Secondly, the potential for multiple equilibria is reduced with their existence becoming more unlikely the more uncertainty there is, i.e. if a higher debt level (and/or cost of levying taxes) does not lead to a stepwise increase in the probability of default.

A higher debt level, or a higher cost of levying taxes, will in general increase the probability that the default option is chosen (the right-hand side of equation (33) is increasing in αD). Thus leading to higher risk premia.

Whether or not there is still potential for multiple equilibria under uncertainty depends on how a higher loss influences the probability of default.

Figure 4 shows the case for three equilibria. The middle (Q2) one is not locally stable in the sense that setting a higher risk premium leads to an even higher premium by investors (and vice versa), but the two outer ones (Q1 and Q3) are locally stable. However, their range is clearly smaller than the range one would obtain under certainty. Moreover, given that they are locally stable it would take a very large shock to shift from one equilibrium to another. For example, if new information about the debt level shifts the curve determining the market interest rate (right hand side of equation (32)) upwards, the economy might enter the region of multiple equilibria (e.g. if the curve shifts from the one depicted in Figure 3 to the one depicted in Figure 4). But in this case the interest rate should increase gradually and the Q1 equilibrium should persist unless a very large shock affects the market. Under certainty, by contrast, which of the two 'sunspot' equilibria will come about is totally indeterminate.

Figure 4. Multiple equilibria under uncertainty
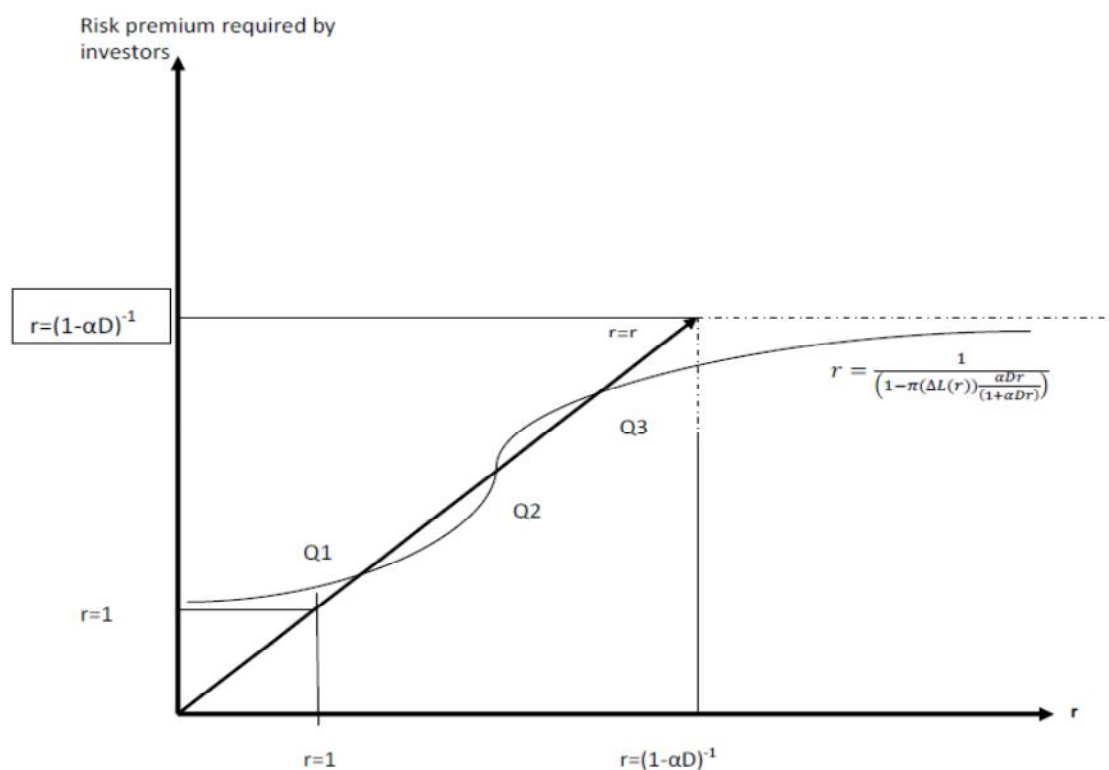
It is worth emphasising that the probability of default that determines the risk premium is the probability as perceived by investors. In a world of efficient markets this should be equal to the objective probability, i.e. ex post the probability that the country actually chooses to default (given the market interest) is equal to the one expected by the market. For those who believe that investors make systematic mistakes it would of course be possible to make a different assumption.

The main insight from this section is that, *ceteris paribus*, a higher subjective expectation of default increases of course the market interest rate and the likelihood that multiple equilibria become possible. But the increase in the interest rate should be gradual, and it remains the case that the equilibrium with the lowest (and the highest) interest rate should be locally stable.

4. Modelling uncertainty explicitly: An illustration

The continued existence of multiple equilibria under uncertainty depends on the precise nature of the stochastic factors that affect the decision to default.

One intuitive and appealing way to think about the lack of automaticity in the default decision is that different parties (and different interest groups) might have different views on the lump-sum cost of default. Which party will win the election cannot be determined with certainty ex ante, but whichever party wins it is clear that the 'temptation' to default will be greater the greater the social loss that can be avoided by defaulting.

It thus seems reasonable to assume that the probability of the government choosing to default increases with the size of the difference between the social loss under default and the loss if debt is serviced in full. One way to formalise this is to think about different parties having different evaluations of the lump-sum cost of defaulting, denoted by LL_d under certainty. Which party wins is a random process since parties compete not only on this issue,

but also on many others. One can then order all parties by their own (subjective) view on LL_d. Given the level of debt, the cost of levying taxes and the interest rate fixed in the market one can then calculate the 'temptation' to default, i.e. the welfare gain from imposing a haircut on creditors and thus lowering the tax burden. If the party that comes to power has a subjective view of the lump-sum cost of defaulting as being smaller than this temptation it will default – and vice versa.

It follows that the probability of default will be given by the value of the cumulative distribution function of the views on LL_d over all parties (or voters) up to the point of temptation.

This function must have the following characteristics: it must be strictly increasing over positive values of the LL_d and it must be bounded by zero (in the vicinity of zero for LL_d) and one (for LL_d going to infinity). A computational convenient functional form would be the following:

$$(36) \quad \pi = \frac{\beta\sqrt[3]{(L_f - L_d)}}{1 + \beta\sqrt[3]{(L_f - L_d)}} = 1 - \frac{1}{1 + r\beta\sqrt[3]{(\alpha^2 D^3(1+\alpha)^{-1})}} = 1 - \frac{1}{1 + rD\beta\theta} < 1 \quad \beta > 0$$

With the composite parameter $\theta(\alpha)$ defined as:

$$(37) \quad \theta \equiv \sqrt[3]{(\alpha^2(1+\alpha)^{-1})}$$

The parameter β indicates the form of the distribution of the views of the lump-sum cost of default over all parties, and thus the sensitivity of the probability of default to the welfare loss from paying in full (compared to the default option). If β is close to zero, the probability of default will always remain close to zero because the views of most parties/voters imply a very high value for LL_d. By contrast a very high value of β represents a situation in which most parties hold the view that the lump-sum cost of default is low, which implies that the probability of default will be high (unless there is either no debt or the cost of levying taxes is zero (D or $\alpha = 0$)).

The functional form for the probability of default in equation (36) implies that the median LLd (i.e. where the probability of default is equal to one half) is given by $LL_{d, median} = rD\beta\theta$. Looking at it from the point of view of the risk premium that will lead to a probability of default equal to one-half, this can be restated as: $r_{\pi=0.5} = LL_{d, median}/D\beta\theta$. Naturally, the higher the debt level the higher the median cost of default must be to maintain the risk premium at the same level.

The determinant of the probability of default can now be substituted back into the equation describing the market equilibrium (33).

$$(38) \quad r = \frac{(1 + \alpha Dr)}{((1 + \alpha Dr) - \pi(\Delta L(r))\alpha Dr)}$$

$$(39) \quad r\big((1 + \alpha Dr) - \pi(\Delta L(r))\alpha Dr\big) = (1 + \alpha Dr)$$

$$(40) \quad \alpha Dr^2[1 - \pi(\Delta L(r))] + r(1 - \alpha D) - 1 = 0$$

Substituting for the value of the probability of default as a function of the risk premium and the other parameters of the models yields:

$$(41) \quad \alpha Dr^2 \left[\frac{1}{1 + rD\beta\theta}\right] + r(1 - \alpha D) - 1 = 0$$

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This can be simplified to a quadratic expression in the risk premium, r.

$$(42) \quad \alpha D r^2 + [r(1 - \alpha D) - 1][1 + rD\beta\theta] = 0$$

$$(43) \quad r^2[\alpha D + (1 - \alpha D)D\beta\theta] + r[(1 - \alpha D) - D\beta\theta] - 1 = 0$$

This expression has clearly only one solution for r in the positive quadrant. This implies that if the probability of default depends on the risk premium in the way specified in equation (36), only one equilibrium exists (whatever the values of all the parameters of the model). It follows that under uncertainty the potential for multiple equilibria can disappear for an entire class of the specification that links the temptation to default to the probability of it actually happening.

5.   Concluding remarks

This paper explores a simple set-up with costs of default, which implies that under certain conditions it might make sense for creditors to forgive part of a country's debt to avoid it choosing to default and thereby paying even less. The simple model presented here also shows that a potential for multiple equilibria can arise.

The framework is useful in analysing the problems in the euro area where governments cannot rely on a central bank to be the buyer-of-last-resort of their debt. Kopf (2011) and De Grauwe (2011) argue that this opens up the potential for multiple equilibria or 'speculative attacks'.

But the model also implies that under the more realistic case of uncertainty the potential for multiple equilibria should be smaller; and that it might actually disappear altogether, depending on the link between expectations of default and some key characteristics of the economy.

Considering the role of uncertainty is essential if models of multiple equilibria are to be used to analyse real world problems, since in reality the decision of a state to default is always preceded by a period of heightened uncertainty during which different groups within the country express widely different positions. The results found here would suggest that the importance of multiple (or sun-spot) equilibria might be overestimated in the crisis affecting the euro area at present.

An underlying assumption in the analysis presented here is that expectations are consistent with the model. It should also be possible to address the consequences of 'irrational' market conditions. Under uncertainty the market rate is determined by the subjective probabilities as perceived by investors. To the extent that these market expectations are higher than the objective ones (i.e. taking into account only the factors driving the decision to default), the risk premium would be higher than one would expect from the fundamentals alone. A 'panic' could lead to higher interest rates and possibly increase the parameter range over which multiple equilibria are possible. This might be a useful avenue for future research.

References

Adrian, T. and D. Gros (1999), "A stochastic model of self-fulfilling crises in fixed exchange rate systems", *International Journal of Finance & Economics*, Vol. 4, No. 2, pp. 129-46.

Borensztein, E. and U. Panizza (2008), "The Costs of Sovereign Default", IMF Working Paper WP/08/238, International Monetary Fund, Washington, D.C.

Calvo, G.A. (1988), "Servicing the Public Debt: The Role of Expectations", *American Economic Review*, Vol. 78, No. 4, September, pp. 647-661.

_____ (1989), "Is inflation effective in liquidating short –term nominal debt?", IMF staff Paper, Vol. 36, No. 4, International Monetary Fund, Washington, D.C.

Daniel, B.C. and C. Shiamptanis (2010), "Sovereign default risk in a Monetary Union", Central Bank of Cyprus Working Paper Series 2010-3.

De Grauwe, P. (2011), "Governance of a Fragile Eurozone", CEPS Working Document No. 346, CEPS, Brussels, May (http://www.ceps.eu/book/governance-fragile-eurozone).

Eichengreen, B., A.K. Rose and C. Wyplosz (1995), "Exchange Market Mayhem: The Antecedents and Aftermath of Speculative Attacks", *Economic Policy*, Vol. 21, pp. 249-312.

Kopf, C. (2011), "Restoring financial stability in the euro area", CEPS Policy Brief No. 237, CEPS, Brussels, March (http://www.ceps.eu/book/restoring-financial-stability-euro-area).

Krugman, P. (1988), "Market-based Debt-reduction schemes", NBER Working Paper No. 2587, National Bureau of Economic Research, Cambridge, MA, May.

Obstfeld, M. (1986), "Rational and Self-fulfilling Balance of Payments Crises", *American Economic Review*, Vol. 76, pp. 72-81.

_____ (1995), "Models of Currency Crises with Self-Fulfilling features", NBER Working Paper No. 5285, National Bureau of Economic Research, Cambridge, MA, October.

Sachs, J., A. Tornell and A. Velasco (1995), "The collapse of the Mexican Peso: What have we learned?", NBER Working Paper No. 5142, National Bureau of Economic Research, Cambridge, MA, June.

Tirole, Jean (2012) "Country Solidarity, Private Sector Involvement and the Contagion of Sovereign Crises" mimeo, 18 June.

Yue, V.Z. (2010), "Sovereign default and debt renegotiation", *Journal of International Economics*, Vol. 80, No. 2, pp. 176-187.

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