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THERMAL NEUTRON SPECTRUM IN A MEDIUM WITH TWO DIFFERENT TEMPERATURES

by H. TAKAHASHI

1962

JOINT NUCLEAR RESEARCH CENTER
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Reactor Physics Department

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Joint Nuclear Research Center.
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Brussels, September 1962 - pages 15 - fig. 3.

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$$T_{eff} = \frac{\mu_1\Sigma_1 T_1 + \mu_2\Sigma_2 T_2}{\mu_1\Sigma_1 + \mu_2\Sigma_2}$$

If the mass ratio μ approaches 1, the distribution deviates from the Maxwellian distribution.

By means of the perturbation method, the simple expression for the deviated flux distribution is obtained. The more rigorous calculation is carried out by expanding the flux in terms of Laguerre polynomials of the first order of energy. This expansion method was used for the study of time and space-dependent problem (3). The generating functions for the matrix element of scattering kernel expanded by the Laguerre polynomials, whose variable E is normalized by a temperature other than the components' temperature, are calculated for the free gas and the crystalline material.

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SUMMARY

The energy-dependent equation in the heavy gas model with the first order correction of μ is considered for the case of a non-absorbing homogeneously mixed medium, whose components have the absolute temperatures T_1 and T_2, the mass ratio μ_1, μ_2 (neutron mass to the mass of the moderator atom) and the macroscopic cross section Σ_1, Σ_2. In the heavy gas approximation, the neutron spectrum distribution is the Maxwellian distribution with the effective temperature defined by:

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The results obtained from the above two methods show that when the moderator components have the same mass, the neutron spectra are shifted to lower energy than the Maxwellian distribution calculated by the heavy gas approximation.

1 — INTRODUCTION

In a power reactor analysis, the neutron spectrum is important for determining a reactivity and a long term reactivity change, etc. The ORGEL-type reactor which is planned in our laboratory is composed of a hot fuel assembly and a cold heavy-water moderator. As a preliminary study of the neutron spectrum in this reactor type, we studied the neutron spectrum in a homogeneous medium whose components have two different temperatures.

When neutrons are put into a non-absorbing infinite medium with temperature T, the neutron spectrum approaches the Maxwellian distribution with temperature T. In the case of a non-absorbing homogeneous medium, whose components have two different temperatures T_1, T_2, the neutron spectrum approaches a distribution lying between the two Maxwellian distributions with T_1 and T_2.

By measuring the neutron spectrum in the homogeneous medium, a study will be made of the effect of the chemical binding of the moderator atom on the neutron spectrum. The neutron spectrum which deviates from the Maxwellian distribution due to the use of an additional absorber has been measured for this kind of study. Using the homogeneous medium whose components have two different temperatures, this effect will be studied without sacrificing the neutron intensity.

In this paper, a medium composed of free gases is studied by using a heavy gas approximation with a first order correction of the mass ratio μ. If a perturbation method

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is used, a simple expression for a neutron spectrum distribution can be obtained as a function of mass ratios, scattering cross sections and component temperatures. The more accurate formalism is obtained in the case of free gas and a crystalline material. The case of free gas with mass 1 is numerically calculated and is compared with the result obtained by the perturbation method.

2 — FORMALISM

In the non-absorbing medium which is composed of two atom components, the neutron balance equation is expressed by

$$\begin{aligned}&[\Sigma_{s1}(E)+\Sigma_{s1}(E)]\,\Phi(E)\\&=\int_0^\infty(\Sigma_{s1}(E'\to E)+\Sigma_{s2}(E'\to E))\Phi(E')\,dE'\end{aligned}\tag{1}$$

where $\Sigma_s(E)$, $\Sigma_s(E'\to E)$ are the scattering cross section and the differential cross section respectively.

In the heavy gas approximation corrected in the first order of μ, the neutron balance equation in a medium whose components have two different temperatures T_1 and T_2, the mass ratios of a neutron to the component atom μ_1 and μ_2 is expressed by the following equation

$$\begin{aligned}\sum_{i=1}^{2}\mu_i\Sigma_i\Bigg[&\left\{\left(ET_i\frac{d^2\Phi}{dE^2}+E\frac{d\Phi}{dE}+\Phi\right)\right.\\&+\mu_i\left\{\frac{4}{3}E^2T_i^2\frac{d^4\Phi}{dE^4}+\frac{8}{3}(T_iE^2+T_i^2E)\frac{d^3\Phi}{dE^3}\right.\\&+\left(\frac{4}{3}E^2+6ET_i+T_i^2\right)\frac{d^2\Phi}{dE^2}\\&\left.\left.+\left(\frac{10}{3}E+T_i-\frac{T_i^2}{E}\right)\frac{d\Phi}{dE}+\left(\frac{2}{3}+\frac{T_i^2}{E^2}\right)\Phi\right\}\right]=0\end{aligned}\tag{2}$$

where Φ is the neutron flux per unit energy interval.

The first terms proportional to μ represent the so-called heavy gas approximation, and the second terms proportional to μ^2 represent the correction for it. In order to solve this equation by a perturbation method, let us consider the terms proportional to μ^2 as perturbation, and write:

$$\Phi(E)=\Phi^{(0)}(E)+\Phi^{(1)}(E)\tag{3}$$

where $\Phi^{(0)}(E)$, $\Phi^{(1)}(E)$ are an unperturbed solution and the perturbed term of the 1st order respectively. From the usual perturbation theory, we get the following equations:

$$\sum_{i=1}^{2} \mu_i \Sigma_i \left\{ E T_i \frac{d^2\Phi^{(0)}}{dE^2} + E \frac{d\Phi^{(0)}}{dE} + \Phi^{(0)} \right\} = 0 \tag{4}$$

$$\sum_{i=1}^{2} \mu_i \Sigma_i \left[\left\{ E T_i \frac{d^2\Phi^{(1)}}{dE^2} + E \frac{d\Phi^{(1)}}{dE} + \Phi^{(1)} \right\} \right.$$

$$+ \mu_i \left\{ \frac{4}{3} E^2 T_i^2 \frac{d^4\Phi^{(0)}}{dE^4} + \frac{8}{3} (T_i E^2 + T_i^2 E) \frac{d^3\Phi^{(0)}}{dE^3} \right.$$

$$+ \left(\frac{4}{3} E^2 + 6 E T_i + T_i^2 \right) \frac{d^2\Phi^{(0)}}{dE^2}$$

$$\left. \left. + \left(\frac{10}{3} E + T_i - \frac{T_i^2}{E} \right) \frac{d\Phi^{(0)}}{dE} + \left(\frac{2}{3} + \frac{T_i^2}{E^2} \right) \Phi^{(0)} \right\} \right] = 0 \tag{5}$$

The equation (3) is rewritten as follows:

$$E \left(\frac{\sum_{i=1}^{2} \mu_i \Sigma_i T_i}{\sum_{i=1}^{2} \mu_i \Sigma_i} \right) \frac{d^2\Phi^{(0)}}{dE^2} + E \frac{d\Phi^{(0)}}{dE} + \Phi^{(0)} = 0$$

and the unperturbed solution $\Phi^{(0)}$ is given by the following Maxwellian distribution:

$$\Phi^{(0)} = \frac{4}{\sqrt{\pi}} \frac{E}{T_{eff}^2} e^{-\frac{E}{T_{eff}}} \tag{6}$$

where

$$T_{eff} = \frac{\sum_{i=1}^{2} \mu_i \Sigma_i T_i}{\sum_{i=1}^{2} \mu_i \Sigma_i}$$

From now on this effective temperature T_{eff} will be expressed by T throughout this paper.

In the heavy gas approximation, the neutron spectrum becomes the Maxwellian distribution with T. And, if one of the moderator components has infinite mass, i.e. $\mu = 0$, and the energy change by scattering is zero, the effective temperature T becomes the temperature of another moderator component.

By substituting equation (6) for equation (5) and rearranging it, we get:

$$\left(T \frac{d^2\Phi^{(1)}}{dE^2} + E \frac{d\Phi^{(1)}}{dE} + \Phi^{(1)} \right)$$

$$= -\frac{1}{\sum_{i=1}^{2} \Sigma_i \mu_i} \sum_{i=1}^{2} \Sigma_i \mu_i^2 \left\{ \frac{4}{3} (T_i - T) \frac{E^3}{T^3} - 2 (4 T_i - 3 T) \frac{E^2}{T^2} \right.$$

$$+(9\,T_i-4\,T)\frac{E}{T}-T_i\Big\}\left(\frac{T_i}{T}-1\right)\ \frac{4}{\sqrt{\pi}}\ \frac{1}{T^2}\,e^{-\frac{E}{T}} \tag{7}$$

Now, in order to solve equation (7), let us expand $\Phi^{(1)}$ and one term of the right hand side of equation (7) by the Laguerre polynomial of 1st order as follows:

$$\Phi^{(1)}(E)=\sum_{m=0}^{\infty} n_m^{(1)}\,L_m^{(1)}\left(\frac{E}{T}\right)\frac{4}{\sqrt{\pi}}\ \frac{E}{T^2}\,e^{-\frac{E}{T}} \tag{8}$$

$$\cdot\left[\frac{4}{3}(T_i-T)\,\frac{E^3}{T^3}-2(4\,T_i-3\,T)\frac{E^2}{T^2}+(9\,T_i-4\,T)\frac{E}{T}-T_i\right]$$

$$=\left[T_i\ \sum_{m=0}^{\infty} a_m\,L_m^{(1)}\left(\frac{E}{T}\right)+T\ \sum_{m=0}^{\infty} b_m\,L_m^{(1)}\left(\frac{E}{T}\right)\right]\left(\frac{E}{T}\right) \tag{9}$$

where a_m and b_m are obtained by using the generating function for the series of the Laguerre polynomials:

$$\frac{e^{-\frac{xt}{1-t}}}{(1-t)^2}=\sum_{m=0}^{\infty} L_m^{(1)}(x)\,t^k\ \ldots\ (t<1) \tag{10}$$

Let us consider the following integration:

$$\int_0^{\infty}\frac{e^{-\frac{xt}{(1-t)}}}{(1-t)^2}\,e^{-x}\left(\frac{4}{3}x^3-8\,x^2+9\,x-1\right)dx$$

$$=8(1-t)^2-16\,(1-t)+9-\frac{1}{(1-t)}$$

$$=-t+7\,t^2-t^3-t^4-t^5-\ldots\ldots \tag{11}$$

On the other hand, from equation (8) this expression is equal to:

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} a_m\int_0^{\infty} L_m^{(1)}(x)\,L_k^{(1)}(x)\,x\,e^{-x}\,dx\,t^k \tag{12}$$

$$=\sum_{m=0}^{\infty} a_m(m+1)\,t^m$$

We get:

$$\begin{aligned} a_0&=0\\ a_1&=-\frac{1}{2}\\ a_2&=\frac{7}{3}\\ a_3&=-\frac{1}{4}\\ a_4&=-\frac{1}{5}\\ &\cdots\cdots \end{aligned} \tag{13}$$

but the convergence of the summation of the "a_m" terms is very slow. Similarly, we get the b_m as follows:

$$\begin{aligned} b_0 &= b_3 = b_4 = \ldots = 0 \\ b_1 &= \frac{1}{2} \\ b_2 &= -\frac{8}{3} \end{aligned} \tag{14}$$

By substituting equations (8) and (9) to (7) and using the orthogonality relation of the Laguerre polynomial, $\Phi^{(1)}$ is obtained as follows:

$$\Phi^{(1)}(E) = \frac{1}{\sum_{i=1}^{2} (\mu_i \Sigma_i)} \sum_{m=0}^{\infty} \sum_{i=1}^{2} (\mu_i^2 \Sigma_i) [T_i a_m + T b_m] \left(\frac{T_i}{T} - 1\right)$$

$$\cdot \frac{1}{m} L_m \left(\frac{E}{T}\right) \frac{4}{\sqrt{\pi}} \frac{E}{T^2} e^{-\frac{E}{T}} \tag{15}$$

The convergence of the sum of the a_m terms in this equation is also very slow. Therefore the same technique which was used for obtaining a_m and b_m is applied for obtaining their summation.

Now, we assume that the sum of the a_m terms is obtained as follows:

$$\sum_{m=0}^{\infty} \frac{a_m}{m} L_m(x)\, x = f(x) \tag{16}$$

By using equations (16) and (10), we get:

$$\begin{aligned} &\int_0^{\infty} \frac{e^{-xt/(1-t)} \cdot e^{-x}}{(1-t)^2} \cdot f(x) dx \\ &= \sum_{m=0}^{\infty} a_m \frac{(m+1)}{m} t^m \\ &= -t + \frac{7}{2} t^2 - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} \cdots \\ &= 4 - 8(1-t) + 4(1-t)^2 - \log(1-t) \end{aligned} \tag{17}$$

From this equation, $f(x)$ is easily obtained by an inverse Laplace transformation, that is:

$$f(x) = 4x - 4x^2 + \frac{2}{3} x^3 - [x \Psi(2) - x \log x] \tag{18}$$

where $\Psi(x)$ is the Euler's Psi function.

In a similar way, we get for the b_m terms:

$$g(x)=\sum_{m=0}^{\infty} b_m \frac{L_m(x)}{m} x = 2x^2 - \frac{2}{3}x^3 \tag{19}$$

Thus, the neutron spectrum distribution is:

$$N(E)=\frac{4}{\sqrt{\pi}}\frac{E}{T^2}\, e^{-\frac{E}{T}} \left[1+\frac{1}{\sum_{i=1}^{2} \Sigma_i \mu_i} \sum_{i=1}^{2} \Sigma_i \mu_i^2 \left(\frac{T_i}{T}-1\right)\left\{\left(\frac{T_i}{T}\right)\frac{f\left(\frac{E}{T}\right)}{\frac{E}{T}}+\frac{g\left(\frac{E}{T}\right)}{\frac{E}{T}}\right\}\right] \tag{20}$$

Now, if we substitute the following ratio for the values of two components, thus obtaining equation:

$$l=\frac{T_2}{T_1}, \qquad m=\frac{\mu_2}{\mu_1}, \qquad n=\frac{\Sigma_2}{\Sigma_1} \tag{21}$$

we get:

$$\Phi(E)=\frac{4}{\sqrt{\pi}}\frac{E}{T^2}\, e^{-\frac{E}{T}} \left[1+\mu_1(l-1)m\,n \cdot \left\{\frac{(ml-1)}{(1+lmn)^2}\frac{f\left(\frac{E}{T}\right)}{\frac{E}{T}}+\frac{(m-1)}{(1+lmn)(1+mn)}\frac{g\left(\frac{E}{T}\right)}{\frac{E}{T}}\right\}\right] \tag{22}$$

where

$$T=T_1\left(\frac{1+lmn}{1+mn}\right) \tag{23}$$

In the case of $l=1$, i.e., when the two components have the same temperature, and when m or n is equal to zero or infinity, the correction factor vanishes and the neutron spectrum becomes Maxwellian. When m equals zero or infinity, the energy change due to neutron scattering by either component is negligible, and when n equals zero or infinity, the neutron scattering by either component is negligible. Thus, the neutron spectrum distribution becomes the Maxwellian distribution of other components.

In the case of $m=1$, i.e., when the masses of the two components are the same, the correction term proportional to $g\left(\frac{E}{T}\right)$ vanishes, and if $lm=1$ is satisfied the correction term proportional to $f\left(\frac{E}{T}\right)$ vanishes. These correction terms proportional to $f\left(\frac{E}{T}\right)$ and $g\left(\frac{E}{T}\right)$ have maximum coefficients in the case of $n=\frac{1}{lm}$ and $n=\frac{1}{m\sqrt{l}}$, respectively, for fixed values of l and m.

So far, the neutron spectrum for the free gas has been calculated in the heavy gas model corrected in a first order of μ by a perturbation method. However, this is not a good solution for the light atom. A more accurate spectrum is obtained in this section by using the expansion of scattering kernel in terms of a orthogonal set of Laguerre polynomial of energy in the 1st order, which was used in the analysis of the space-dependent problem in paper I(3). i.e. it is assumed that the fluxes Φ in the equation (1) are expanded in the following way:

$$\Phi(E)=\sum_{i=0}^{\infty}\frac{A_i}{\sqrt{(i+1)}}L_i\left(\frac{E}{T}\right)\frac{E}{T^2}\,e^{-\frac{E}{T}} \tag{24}$$

substituting eq (24) into eq (1) and multiplying the resulting equation by $\frac{1}{\sqrt{(j+1)}}L_j\left(\frac{E}{T}\right)$ and integrating over E, we get

$$[S_{ij1}+S_{ij2}]\,A_i=0 \tag{25}$$

where

$$S_{ijk}=\frac{1}{\sqrt{(i+1)(j+1)}}\left[\int_0^{\infty}\int_0^{\infty}dE'\,dE\Sigma_{sk}(E'\to E)L_i^{(1)}\left(\frac{E'}{T}\right)L_j^{(1)}\left(\frac{E}{T}\right)\frac{E'}{T^2}\,e^{-\frac{E'}{T}}\right.$$

$$\left.-\int_0^{\infty}\int_0^{\infty}dE\,dE'\,\Sigma_{sk}(E'\to E)L_i^{(1)}\left(\frac{E'}{T}\right)L_j^{(1)}\left(\frac{E'}{T}\right)\frac{E'}{T^2}\,e^{-\frac{E'}{T}}\right] \tag{26}$$

In this case, the temperature used in the expansion of flux equation (26) is different from the component temperatures, since the generating function S can be calculated in the same way as the case of the space-dependent problem in the paper I(3). The results are shown in the following. In the case of free gas, we get:

$$S_k=-4\Sigma_B\frac{m}{M_k}\frac{T_k}{T}\frac{\left[\left(1-\frac{T}{T_k}\right)+\frac{T}{T_k}P\right]l}{(1-Pl)^2\left(1+\frac{m}{M_k}\right)^4}\frac{\sqrt{1+\frac{\frac{mT_k}{M_kT}}{1+\frac{mT_k}{M_kT}}\left(\frac{P}{1-P}+\frac{l}{1-l}\right)}}{\left[1+4\frac{T_k}{T}\frac{m}{M_k}\frac{\left(1-\frac{T}{T_k}\right)}{\left(1+\frac{m}{M_k}\right)^2}l-\frac{\left(1-\frac{m}{M_k}\right)^2}{\left(1+\frac{m}{M_k}\right)^2}Pl\right]}$$

The generating function S in the case of crystalline material is obtained by the mass expansion method as follows

$$S_k=-2\Sigma_B\frac{(1-P)^{-1}(1-l)^{-1}}{(1-Pl)}\sum_{q=1}^{\infty}\left(-\frac{m}{M_k}K_k\right)^q\sum_{n=1}^{q}\frac{1}{\Gamma(q-n+1)\Gamma(n+1)}\int_0^{\omega_m}\sinh$$

$$\left(\left(\frac{P}{1-P}+1-\frac{T}{T_k}\right)\zeta\right)\sinh\left(\frac{l}{1-l}\zeta\right)G_n'(\zeta)\zeta^{q+1}K_{q+1}\left(\left(1+\frac{P}{1-P}+\frac{1}{1-l}\right)\zeta\right)d\omega \tag{28}$$

and by the Phonon expansion method, we get:

$$S_k = -2\Sigma_B \frac{(1-P)^{-1}(1-l)^{-1}}{(1-Pl)} \sum_{q=1}^{\infty} \left(\frac{m}{M_k}\right)^q \frac{1}{q!} \int_0^{\omega_m} \sinh\left(\left(\frac{P}{1-P}+1-\frac{T}{T_k}\right)\zeta\right)$$

$$\sinh\left(\frac{1}{1-l}\zeta\right) G_q''(\zeta)\zeta^{q+1} \frac{K_{q+1}\left(\gamma\left(1+\frac{P}{1-P}+\frac{l}{1-l}\right)\zeta\right)}{\gamma^{(q+1)}} d\omega \qquad (29)$$

In these equations the same notations are used as in paper *I*. We can also get the generating function by using the two Phonon dividing models. The matrix elements $S_{ij}^{(i)}$ are obtained as the coefficients of $\sqrt{(i+l)(j+1)}\,P^i l^j$ terms.

Let us consider the case of heavy gas, that is $\frac{m}{M} \to 0$, the generating function of equation (27) becomes

$$\sum_{k=1}^{2} S_k = -4 \sum_{k=1}^{2} \Sigma_{Bk} \frac{m}{M_k} \frac{T_k}{T} \frac{\left[\left(1-\frac{T}{T_k}\right)+\frac{T}{T_k}l\right]l}{(1-Pl)^3} \qquad (30)$$

If we choose the temperature T in such a way that

$$\sum_{k=1}^{2} \frac{\Sigma_{Bk}}{M_k}\left(\frac{T_k}{T}-1\right) = 0 \qquad (31)$$

we get

$$\sum_{k=1}^{2} S_k = -4 \sum_{k=1}^{2} \Sigma_{Bk} \frac{m}{M_k} \frac{Pl}{(1-Pl)^3} \qquad (32)$$

which is the generating function in the case of heavy gas with

$$\sum_{i=1}^{2} \Sigma_{Bk} \frac{m}{M_k}$$

instead of

$$\Sigma_B \frac{m}{M}$$

and its matrix is diagonalized. Thus, the lowest eigenfunction becomes the Maxwellian distribution with the effective temperature

$$T = \frac{\sum_{k=1}^{2} (\mu_k \Sigma_k T_k)}{\sum_{k=1}^{2} (\mu_k \Sigma_k)} \qquad (33)$$

This is the result obtained in the last section.

3 — NUMERICAL RESULTS

In figures 1, 2 and 3, the neutron spectrum is calculated using equation (22) for the cases where $l = 2.0$ and 3.0 and $\mu_1 = 1.0$, $m = 1$, $n = 1$ for both cases. Also the case of $l = \frac{1}{2}$, $\mu_1 = 1.0$, $m = 2.0$, $n = 1$ is shown. In all cases neutron spectra with effective temperature T, which are calculated from the heavy gas approximation, are also shown as Maxwellian distributions.

In the first two cases, the deviation from the Maxwellian distribution is due to the term proportional to $f\left(\frac{E}{T}\right)$ and the peak of the neutron spectrum is shifted to the lower energies. In the third case, the deviation is only due to the term proportional to $g\left(\frac{E}{T}\right)$ and the peak of the neutron spectrum does not shift appreciably. Furthermore, the case of $\mu_1 = 1$, $l = 2.0$, $m = \frac{1}{3}$, $n = 1.0$ are added in figure 3 where both correction terms (f and g) are included. These deviations decrease as the moderator atom mass is increased.

In table 1, the coefficients of proportional to $f\left(\frac{E}{T}\right)\Big/\frac{E}{T}$ and $g\left(\frac{E}{T}\right)\Big/\frac{E}{T}$

$$C_1 = \frac{(l-1)mn(ml-1)}{(1+lmn)^2}$$

$$C_2 = \frac{(l-1)mn(m-1)}{(1+lmn)(1+mn)}$$

are tabulated for several typical cases.

Table 1 THE VALUE OF C_1 AND C_2

	$\frac{\sigma_2}{\sigma_1}$ / $\frac{T_2}{T_1}$ / $\frac{\mu_2}{\mu_1}$	1		2		3	
		2	3	2	3	2	3
C_1	1	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{8}{25}$	$\frac{3}{25}$	$\frac{1}{3}$
	$\frac{1}{2}$	0	$\frac{2}{25}$	0	$\frac{4}{49}$	0	$\frac{2}{9}$
	$\frac{1}{3}$	$-\frac{1}{25}$	0	$-\frac{1}{32}$	0	$-\frac{3}{121}$	0
C_2	1	0	0	0	0	0	0
	$\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{2}{15}$	$-\frac{1}{15}$	$-\frac{4}{35}$	$-\frac{3}{56}$	$-\frac{2}{21}$
	$\frac{1}{3}$	$-\frac{1}{10}$	$-\frac{1}{6}$	$-\frac{1}{14}$	$-\frac{8}{63}$	$-\frac{3}{55}$	$-\frac{1}{10}$

In order to compare the spectrum obtained by the expansion method with the spectrum obtained by an approximate method, the neutron spectra in the free gas medium with mass 1 are calculated in the case of $T_2/T_1 = 2$ and 3. Their spectra are shown in figures 1, 2 with the numerical results calculated by the perturbation method.

Their deviations from the Maxwellian distribution of the heavy gas approximation are smaller than the deviation which results from the perturbation method. In the medium which is composed of atoms with the same mass, we find that the energies where the maximum of neutron spectrum is located deviate to an energy lower than the one calculated by the heavy gas approximation.

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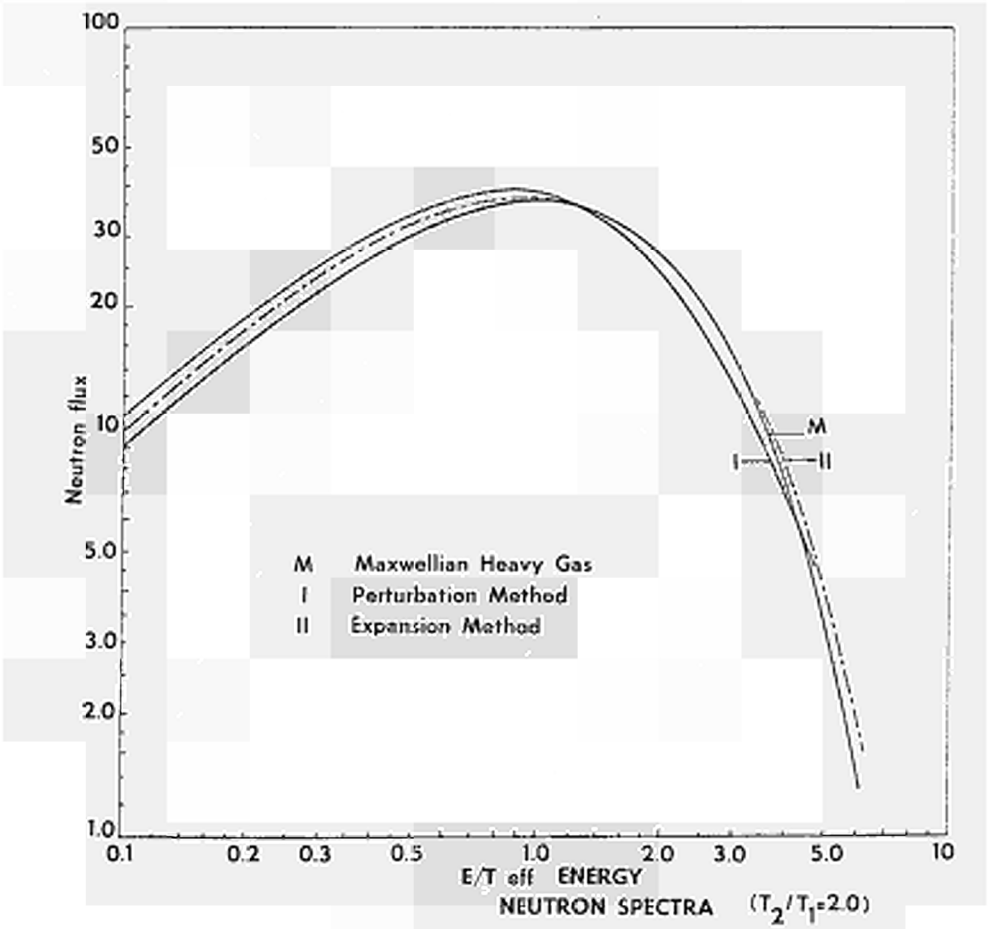
100
50
30
20
10
5.0
3.0
2.0
1.0
Neutron flux
M
I
II
M Maxwellian Heavy Gas
I Perturbation Method
II Expansion Method
0.1
0.2
0.3
0.5
1.0
2.0
3.0
5.0
10
E/T eff ENERGY
NEUTRON SPECTRA (T_2/T_1=2.0)

FIG. 1.

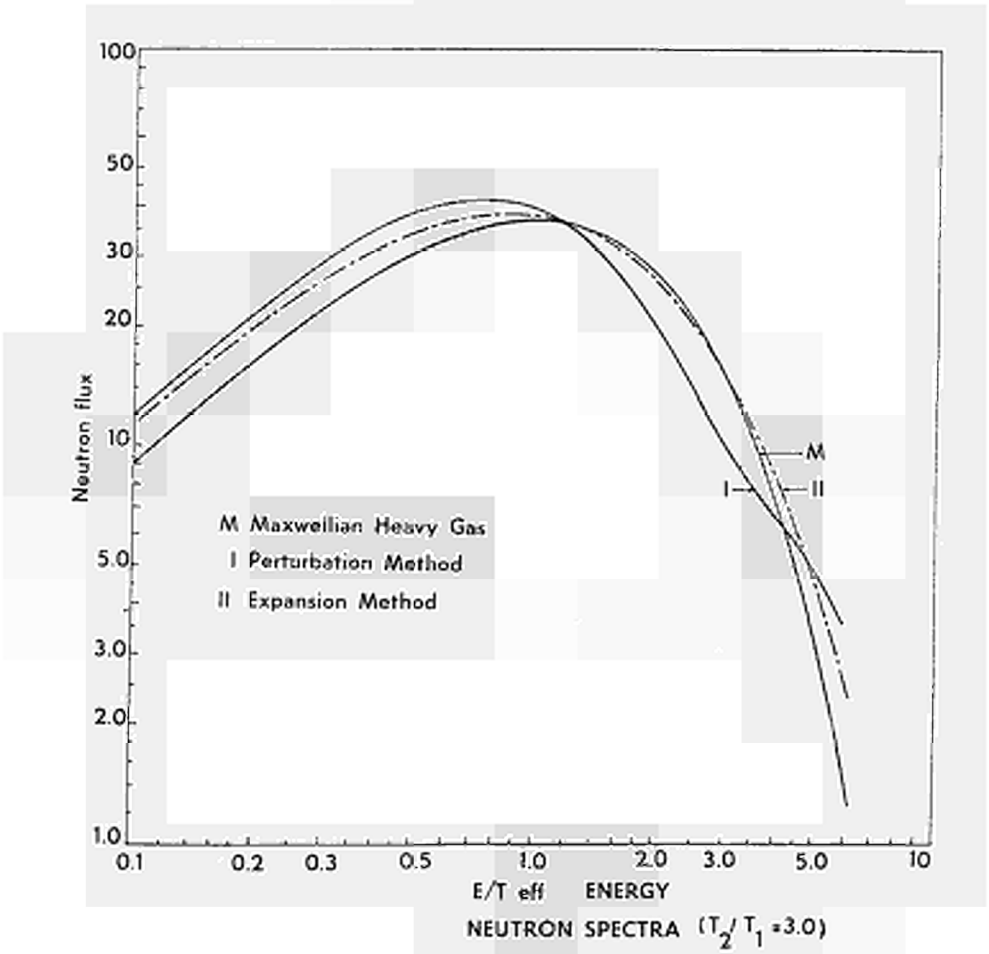
100
50
30
20
10
5.0
3.0
2.0
1.0
Neutron flux
M
I
II
M Maxwellian Heavy Gas
I Perturbation Method
II Expansion Method
0.1
0.2
0.3
0.5
1.0
2.0
3.0
5.0
10
E/T eff ENERGY
NEUTRON SPECTRA (T_2/T_1=3.0)

FIG. 2.

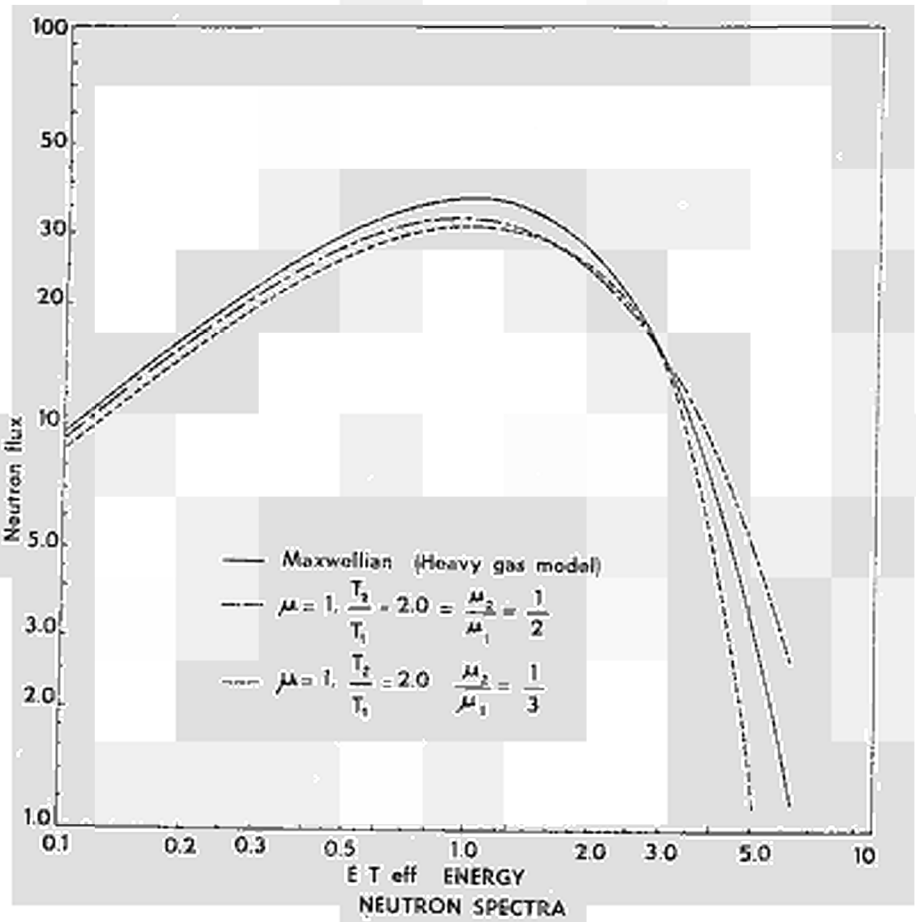
100
50
30
20
10
5.0
3.0
2.0
1.0
Neutron flux
0.1
0.2
0.3
0.5
1.0
2.0
3.0
5.0
10
E T eff ENERGY
NEUTRON SPECTRA
Maxwellian (Heavy gas model)
μ = 1, T2/T1 = 2.0 = μ2/μ1 = 1/2
μ = 1, T2/T1 = 2.0 μ2/μ1 = 1/3

FIG. 3.

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